Optimal combination of quaternions from multiple star cameras

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The purpose of this note is to sketch a procedure for optimally combining quaternions measured simultaneously from differently aligned star cameras, given (Gaussian) noise models referred to the individual star camera frames. The application to the GRACE mission (with two cameras) will be described in detail.

We consider a spacecraft with multiple star cameras labelled by $\alpha = 1, 2, \ldots, N$. The cameras are assumed to be rigidly fixed with respect to each other, so that the rotations $Q_{\alpha\to\beta}$ between the cameras can be well determined from long time series of data. We take as given a common fiducial frame $\mathcal{C}$, fixed with respect to all the cameras. $\mathcal{C}$ might coincide with one of the cameras, or with some body-fixed frame whose tie to the ensemble of cameras is somehow determined. At any rate, the rotations $Q_{\alpha\to\mathcal{C}}$ from each camera to the common frame $\mathcal{C}$ are assumed to be known.

Given the N measured quaternions

$$Q^{\text{meas}}_{\mathcal{I}\to\alpha} \tag{1}$$

from inertial space $\mathcal{I}$ to the individual frames α, the problem is to combine them into an optimal estimate

$$Q^{\text{opt}}_{\mathcal{I}\to\mathcal{C}} \tag{2}$$

of the rotation from $\mathcal{I}$ to the common frame $\mathcal{C}$. Each measured quaternion (1) is modelled as the product of a "true" quaternion and a "noise" quaternion:

$$Q^{\text{meas}}_{\mathcal{I}\to\alpha} \;=\; Q^{\text{true}}_{\mathcal{I}\to\alpha} \cdot Q^{\text{noise}}_{\alpha} \tag{3}$$

The noise quaternion is assumed to be close to unity:

$$Q^{\text{noise}}_{\alpha} \;=\; (1, \tfrac{1}{2}\epsilon_{\alpha}) + \mathcal{O}(\epsilon^2) \tag{4}$$

where the three "vector" components $\epsilon_{\alpha,i}$ $(i = x, y, z)$ are small, treated as infinitesimal (*i.e.,* our analysis is consistently linearized about them). The factor of $\frac{1}{2}$ is so the ϵ's can be interpreted as angles.

From the order in which the quaternions are written in (3), the components of ϵ_{α} are referred to the star-camera frame α (as opposed to the inertial frame $\mathcal{I}$). It is assumed that the components obey Gaussian statistics

$$\left\langle \epsilon_{\alpha,i}\epsilon_{\alpha,j} \right\rangle \;=\; \left(C_{\alpha}\right)_{ij} \tag{5}$$

where the components of the covariance matrix C_{α} are known in the star-camera frame α.

Define

$$Q^{\text{meas}(\alpha)}_{\mathcal{I}\to\mathcal{C}} \;\equiv\; Q^{\text{meas}}_{\mathcal{I}\to\alpha} \cdot Q_{\alpha\to\mathcal{C}} \tag{6}$$

which is essentially the estimate of the desired rotation (2) from each separate camera. These N quaternions should be close to each other. (If some are close to negatives of others, the arbitrary signs in the individual measurements should be fixed to remedy the sign descrepancies.) Then the offsets between pairs are

$$\begin{aligned}
(Q^{\text{meas}(\alpha)}_{\mathcal{I}\to\mathcal{C}})^{-1} \cdot (Q^{\text{meas}(\beta)}_{\mathcal{I}\to\mathcal{C}}) &= Q_{\mathcal{C}\to\alpha} \cdot (1, -\tfrac{1}{2}\epsilon_\alpha) \cdot Q^{\text{true}}_{\alpha\to\mathcal{I}} \cdot Q^{\text{true}}_{\mathcal{I}\to\beta} \cdot (1, \tfrac{1}{2}\epsilon_\beta) \cdot Q_{\beta\to\mathcal{C}} \\
&= (1,0) \;-\; Q_{\mathcal{C}\to\alpha} \cdot (0, \tfrac{1}{2}\epsilon_\alpha) \cdot Q_{\alpha\to\mathcal{C}} \;+\; Q_{\mathcal{C}\to\beta} \cdot (0, \tfrac{1}{2}\epsilon_\beta) \cdot Q_{\beta\to\mathcal{C}} \\
&\equiv (1, \tfrac{1}{2}\Delta_{\alpha\beta})
\end{aligned} \tag{7}$$

(to first order in the ϵ's), where we have defined

$$\Delta_{\alpha\beta} = \tilde{\epsilon}_\beta - \tilde{\epsilon}_\alpha \tag{8}$$

in terms of the ϵ's (considered as vectors) rotated into the common frame $\mathcal{C}$:

$$\tilde{\epsilon}_\alpha \equiv R_{\alpha\to\mathcal{C}}\epsilon_\alpha. \tag{9}$$

We have used the composition of quaternions:

$$Q^{\text{true}}_{\alpha\to\mathcal{I}} \cdot Q^{\text{true}}_{\mathcal{I}\to\beta} = Q_{\alpha\to\beta} = Q_{\alpha\to\mathcal{C}} \cdot Q_{\mathcal{C}\to\beta} \tag{10}$$

and the consistency relation (easily verified, using the standard definitions) between quaternion rotations and rotation matrices acting on vectors:

$$Q_{\mathcal{C}\to\alpha} \cdot (0, \epsilon_\alpha) \cdot Q_{\alpha\to\mathcal{C}} = (0, R_{\alpha\to\mathcal{C}}\epsilon_\alpha) \tag{11}$$

The equations (7) in fact provide $(N-1)$ independent linear relations between the N unknowns ϵ_α. The effect is to allow $(N-1)$ of the ϵ_α to be determined in terms of the remaining one (say, the first one):

$$\tilde{\epsilon}_\alpha = \Delta_{1\alpha} + \tilde{\epsilon}_1 \tag{12}$$

where the Δ's are calculated from the measured quaternions according to (7).

The optimal solution is then determined by minimizing the cost functional corresponding to the statistics (6):

$$J \equiv \sum_\alpha \epsilon_\alpha^{\mathrm{T}} \Lambda_\alpha \epsilon_\alpha \tag{13}$$

where the information matrices are the inverses of the covariance matrices:

$$\Lambda_\alpha \equiv (C_\alpha)^{-1} \tag{14}$$

Inserting (12) into (13) gives

$$J = \tilde{\epsilon}_1^{\mathrm{T}} \widetilde{\Lambda}_1 \tilde{\epsilon}_1 + \sum_{\alpha\neq 1} (\tilde{\epsilon}_\alpha^{\mathrm{T}} + \Delta_{1\alpha}^{\mathrm{T}}) \widetilde{\Lambda}_\alpha (\tilde{\epsilon}_\alpha + \Delta_{1\alpha}) \tag{15}$$

where we define the information matrices rotated into the common frame $\mathcal{C}$, and their sum:

$$\begin{aligned} \widetilde{\Lambda}_\alpha &\equiv R_{\alpha\to\mathcal{C}} \Lambda_\alpha R_{\mathcal{C}\to\alpha} \\ \widetilde{\Lambda}_{\text{tot}} &\equiv \sum_\alpha \widetilde{\Lambda}_\alpha \end{aligned} \tag{16}$$

One easily finds that the functional J is extremized for

$$\tilde{\epsilon}_1^{\text{opt}} = -\widetilde{\Lambda}_{\text{tot}}^{-1} \sum_{\alpha\neq 1} \widetilde{\Lambda}_\alpha \Delta_{1\alpha} \tag{17}$$

giving for the optimal solution:

$$\begin{aligned} Q^{\text{opt}}_{\mathcal{I}\to\mathcal{C}} &= Q^{\text{opt}}_{\mathcal{I}\to 1} \cdot Q_{1\to\mathcal{C}} \\ &= Q^{\text{meas}}_{\mathcal{I}\to 1} \cdot (1, -\tfrac{1}{2}\epsilon_1^{\text{opt}}) \cdot Q_{1\to\mathcal{C}} \\ &= Q^{\text{meas}}_{\mathcal{I}\to 1} \cdot Q_{1\to\mathcal{C}} \cdot (1, -\tfrac{1}{2}\tilde{\epsilon}_1^{\text{opt}}) \\ &= Q^{\text{meas}(1)}_{\mathcal{I}\to\mathcal{C}} \cdot \left(1,\ \tfrac{1}{2}\widetilde{\Lambda}_{\text{tot}}^{-1} \sum_{\alpha\neq 1} \widetilde{\Lambda}_\alpha \Delta_{1\alpha}\right) \end{aligned} \tag{18}$$

A little algebra shows that the result does not depend on the special role played by the first camera.

Application to GRACE

For the GRACE mission, there are two star cameras, mounted (to within a degree or two) at

$$Q_{1;2\to\mathcal{C}} \approx (\cos(135°), \pm\sin(135°)\hat{\mathbf{x}}) \tag{19}$$

where $\mathcal{C}$ is the body-fixed Science Reference Frame. The rotation matrices corresponding to (19) are

$$R_{1;2\to\mathcal{C}} \approx \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/\sqrt{2} & \pm 1/\sqrt{2} \\ 0 & \mp 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \tag{20}$$

These approximate forms will only be used to transform the formal errors.

The star cameras have the nominal noise characteristics

$$C_1 = C_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \kappa^2 \end{bmatrix} \sigma^2 \tag{21}$$

where z is the camera boresight direction, the formal error in the perpendicular directions is about $\sigma \approx 6$ arcsec, and the formal error for rotations about the boresight is a factor of $\kappa \approx 8$ greater.

The $\widetilde{\Lambda}$ matrices, using the approximate rotations between the camera frames and $\mathcal{C}$, are

$$\widetilde{\Lambda}_{1;2} \;=\; \frac{1}{2\sigma^2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & (1+\kappa^{-2}) & \pm(1-\kappa^{-2}) \\ 0 & \pm(1-\kappa^{-2}) & (1+\kappa^{-2}) \end{bmatrix} \tag{22}$$

It should be sufficient to use the approximate rotations to determine these matrices, since anyway the camera covariance matrices are only approximately determined. One then has

$$M_{1;2} \;\equiv\; \widetilde{\Lambda}_{\rm tot}^{-1}\,\widetilde{\Lambda}_{1;2} \;=\; \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \pm\lambda \\ 0 & \pm\lambda & 1 \end{bmatrix} \tag{23}$$

with

$$\lambda \;\equiv\; \frac{(\kappa^2-1)}{(\kappa^2+1)} \tag{24}$$

The procedure for optimal combination is as follows:

(i) In terms of the measurements and the rotations from each camera to the common frame, define according to (6):

$$Q^{\rm meas(1;2)}_{\mathcal{I}\to\mathcal{C}} \;\equiv\; Q^{\rm meas}_{\mathcal{I}\to 1;2}\cdot Q_{1;2\to\mathcal{C}} \tag{25}$$

(ii) Determine the three-dimensional (small) vector Δ_{12} according to (7):

$$(Q^{\rm meas(1)}_{\mathcal{I}\to\mathcal{C}})^{-1}\cdot(Q^{\rm meas(2)}_{\mathcal{I}\to\mathcal{C}}) \;=\; (1,\,\tfrac{1}{2}\Delta_{12}) \tag{26}$$

(iii) The optimal quaternion is given according to (18) and (23)

$$Q^{\rm opt}_{\mathcal{I}\to\mathcal{C}} \;=\; Q^{\rm meas(1)}_{\mathcal{I}\to\mathcal{C}}\cdot\big(1,\,\tfrac{1}{2}M_2\Delta_{12}\big) \tag{27}$$